MATHEMATICAL MODELING OF CAVITATING FLOWS

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ABSTRACT

Many problems of cavitations (e.g. cavitating flows around hydrofoils and bodies of rotation, design of supercavitating foils, ventilated cavities etc.) are in detail enough investigated both experimentally and theoretically. However there are still more many problems, which require more careful research. Some of these problems are considered in the present report.

The basic problem in hydrodynamics is mathematical modeling of physical processes. It is formally possible to write the differential equations, and also initial and boundary conditions. But it is not obviously possible to solve these equations even with use super-computers. Therefore, the certain physical parameters and processes are excluded from consideration by theoretical studying and the simplified mathematical models are investigated. Naturally, such models in accuracy may not reflect real processes; nevertheless, under certain conditions theoretical results well enough may be coordinated with experimental data and enable to receive the detailed characteristic of investigated processes.

In the present report the simplified mathematical models for some problems of cavitations entry into liquid are considered.

INTRODUCTION

Many problems in cavitation (e.g., cavitating flows around hydrofoils and bodies of revolution, design of supercavitating foils, ventilated cavities, etc.) have been investigated in detail both experimentally and theoretically. However, there are still many more problems that require more careful research. Some of these problems are considered in the present report.

The basic problem in hydrodynamics is the mathematical modeling of physical processes. It is possible to write the relevant differential equations and to establish initial and boundary conditions. But it is not possible to solve these equations even using supercomputers. Therefore, certain physical parameters and processes are excluded from consideration in theoretical study and investigation with simplified mathematical models. Naturally, such models lack accuracy and may not reflect real processes; nevertheless, under certain conditions theoretical results may be sufficiently correlated with experimental data to determine the detailed characteristics of various investigated processes.

In the present report simplified mathematical models for some problems of the introduction cavitation into a liquid are considered.

1. A BRIEF ANALYSIS OF CAVITATING MODELS

Actual cavitating flow is unsteady and the cavity boundary is not smooth. Such a process cannot be completely described, theoretically, but fortunately the non-steady-state condition does not have much influence on the whole relative to steady flow and the pressure inside the cavity is about constant. This permits the assumption that the flow is steady and the cavity boundary is smooth. Viscosity has little influence on cavitation because of free cavity boundaries, thus it can be neglected. If the flow is free of vortices, then the cavitating flow can be simulated by the mathematical approach to the problem for a harmonic function where the kinematical condition ($\nu = 0$) is satisfied on the solid and free boundaries and dynamic condition ($\nu = \nu = const$) is satisfied on unknown free boundary. This problem was formulated long ago by Helmholtz (1868); of course, it was not connected with cavity flow. Further, the flow problems will be considered only under the assumption of two-dimensional flow.

Helmholtz’s problem has mixed value conditions and can be solved in classes of both singular and continuous functions. The continuous solution corresponds to the open model of cavitation considered by T. Wu (1964). One kind of singularity is described by Brillion’s condition of minimum pressure in the cavity and mathematically has the form $\ln(\frac{d\nu}{d\zeta}) = \nu^{1/2}$. This singularity was first proposed by M. Tulin (1963) and substantiated mathematically by A.G. Terentiev (1976). The cavity boundaries terminate at the fast converged spirals because of the singularity. At the present time, numerous problems of cavitating flow are being investigated using this model (Terentiev 1981, Maklakov 1997, Dimitrieva, Terentiev 1998, 2002).

Note that the spiral cavitation model appears from the value problem mentioned above to be distinct from other models based on assumptions of various flow conditions at end of the cavity. Fortunately, the nature of the aft stream has little influence on the overall cavitating flow. Some of the first models, as the Riabouchinsky end-plate model or
the re-entrant jet model, allow one to obtain coincident results for hydrodynamic forces and measurement of the cavity. However, these models, to a certain extent, present some difficulties connected with the uncertainty of the end-plate and the direction of the re-entrant jet. The latter will be directed contrary to the stream velocity, but it can be inclined at any angle. Similar uncertainties arise with respect to the end-plate in the Riabouchinsky model.

The cavity shapes for angle of an attack of \( \alpha = \pi / 6 \) and a cavitation number of \( \sigma = 0.5 \) calculated by using both end-plate and single-spirals models are shown in Figure 1. The two models produce a match if the end-plate is perpendicular to the stream velocity. The data position of the centers of the spirals \((x_1=0.391, y_1=0.097)\) and \((x_2=3.934, y_2=0.398)\) are so close to the edges of the end-plate \((3.935, 0.0767\) and \(3.935, 0.418\)) that they cannot be recognized on the figure. The flow forces coincide to an accuracy within four figures; namely, the coefficient of resultant force is \(C_n=0.9039\). Figure 1 shows also the cavity shape for the central symmetric model calculated by A.G. Terentiev (1964). The coefficient of resultant force is \(C_n=0.8924\). The value problem for the last model is relatively easy to solve analytically, but the calculated results for small angles of attack differ from the above-mentioned two models.

### Table 1. Comparison of the lift coefficients for a flat plate calculated by linear approach \( (C_L) \) and non-linear theory \( (C_L) \) for various angles of attack \( (\alpha) \) and cavitation numbers \( (\sigma) \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( C_L^{\text{linear}} )</th>
<th>( C_L^{\text{non-linear}} )</th>
<th>( \sigma )</th>
<th>( C_L^{\text{linear}} )</th>
<th>( C_L^{\text{non-linear}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1251</td>
<td>0.1244</td>
<td>0.1</td>
<td>0.1692</td>
<td>0.1683</td>
</tr>
<tr>
<td>2</td>
<td>0.1218</td>
<td>0.1219</td>
<td>0.2</td>
<td>0.2516</td>
<td>0.2524</td>
</tr>
<tr>
<td>3</td>
<td>0.1320</td>
<td>0.1325</td>
<td>0.3</td>
<td>0.3628</td>
<td>0.3613</td>
</tr>
<tr>
<td>4</td>
<td>0.1485</td>
<td>0.1490</td>
<td>0.4</td>
<td>0.4697</td>
<td>0.4634</td>
</tr>
<tr>
<td>10</td>
<td>0.2908</td>
<td>0.2726</td>
<td>0.5</td>
<td>0.6257</td>
<td>0.6123</td>
</tr>
</tbody>
</table>

The flow on \( z \)-plane is uniquely determined by the complex potential \( w(z) \). The function \( w(z) \) can be found by conformal mapping of the flow region onto half of a unit circle in the auxiliary \( \zeta \)-plane, as shown in Figures 3a and 3b, respectively.

Because of the asymptotic approach, the linear theory determines relatively precise values of the main parameters of the flow (see Table 1); therefore, for engineering purposes it is adequate to use linear theory. But there are many problems that cannot be addressed in the frame of linear theory. Below, we shall consider both linear and non-linear problems of cavitating flow.

## 2. PARTIALLY CAVITATING FLOW

Partially cavitating flows attracted attention at the Third International Symposium on Cavitation. Some important results with numerical calculations had been obtained (Kinnas 1998, Laberteaux, Ceccio 1998 and others). It is important to predict partial cavities on a foil because cavitation erosion occurs mostly at end of the cavity. It appears that the analytic or numerical investigation is not difficult. But it is only partly valid and only if the foil has a special form, for instance, a flat plate or a wedge. For clarification of some problems with partial cavitating flows, a triangular foil is considered (Figure 2).

In applying the re-entrant jet cavitation model, we must choose the direction of the jet. The problem can be solved for both possible directions \( a \) and \( b \) in Figure 2. Of course, the first one is preferable and simple to solve; but the second alternative is also theoretically entitled to an existence. It is more difficult to choose the direction of the jet for a foil with curvilinear surfaces. The same problem arises with the single spiral cavitation model (Figure 3a).

Applying the re-entrance jet model to a flat plate with partial cavitation has been considered by A.V. Kuznezov and A.G. Terentiev (1967). Models with an end-plate and with a single spiral also have been applied to the investigation of partial cavitating flow on a plate and a cascade by A.G. Terentiev (1970, 1981). Below, the last model is solved analytically by conformal mapping of the flow region onto half of a unit circle in the auxiliary \( \zeta \)-plane, as shown in Figures 3a and 3b, respectively.

The flow on \( z \)-plane is uniquely determined by the complex potential \( w(z) \). The function \( w(z) \) can be found by conformal mapping of the flow domain on \( z \)-plane onto
some parametrical region on the auxiliary $\zeta$ - plane. Considering the semicircle (Figure 3b) for the parametrical region (according to the points as shown in Figures 3a and 3b), one can find the derivative $dw/d\zeta = w_\zeta(\zeta)$ and the velocity function $dw/dz = w_\zeta(\zeta)$ by using their singularity and zeros.

Since $1m w$ is constant on the boundary, then the potential $w(\zeta)$ can be extended analytically over the total $\zeta$ - plane. Then its derivative, $w_\zeta(\zeta)$, has simple zero at points $C(\zeta=1)$, $B(\zeta=-1)$, $O(\zeta_0 = e^{i\theta})$, $A(\zeta_a = e^{i\theta})$, $\overline{A}(\overline{\zeta}_a = e^{-i\theta})$, $\overline{C}(\overline{\zeta}_a = a^{-i\theta})$ and second-order poles at points $D(id)$, $D'(i/d)$, $\overline{B}(-id)$ and $\overline{B}'(-i/d)$. Hence,

$$w_{\zeta} = -\frac{(\zeta^2 - 1)(\zeta-e^{i\theta})(\zeta-e^{-i\theta})(\zeta-e^{i\theta})(\zeta-e^{-i\theta})}{(\zeta^2 + a^2)(2[\zeta^2 + 1])}. \quad (1)$$

The complex velocity ($w_{\zeta} = ve^{-i\theta}$) has a constant value ($\nu=1$) on the free boundary ($BC$) and a step argument on the foil, namely:

$$\theta = -\pi \mu \text{ on } CA, \quad \theta = 0 \text{ on } CO, \quad \theta = -\pi \text{ on } OB.$$ 

Therefore, the function $w_{\zeta}(\zeta)$ has a zero at point $O$ and $\mu$-order zero at point $A$. After analytical extension over the total $\zeta$ - plane, the function $w_{\zeta}(\zeta)$ will have singularities at symmetrical points $\overline{O}$ and $\overline{A}$ as $(\zeta-e^{i\theta})^{-1}$ and $(\zeta-e^{-i\theta})^{-\mu}$. Moreover, the logarithmic function ($\ln w_{\zeta}(\zeta)$) has a simple pole at the point $C(\zeta=1)$, so that $w_{\zeta}(\zeta)$ has an exponential singularity, which can be written as:

$$T(\zeta) = \exp\left(-ic \frac{\zeta + 1}{\zeta - 1}\right). \quad (2)$$

Hence,

$$w_{\zeta} = -e^{-i(b+\mu)} \frac{(\zeta-e^{i\theta})(\zeta-e^{i\theta})^\mu}{(\zeta-e^{-i\theta})(\zeta-e^{-i\theta})^\theta} T(\zeta). \quad (3)$$

The factor in Equation (2) is obtained from the condition at point $B$ ($w_{\zeta}(-1) = -1$). Under this condition, all dimensional parameters are related to the value of the velocity on the cavity and to the density of the fluid. Equation (1) relates to a foil, which is similar to the given condition. Therefore, the characteristic length and other geometric parameters should be calculated and then divided by them.

The mapping function can be determined by integration of the function

$$z_\zeta(\zeta) = \frac{w_{\zeta}(\zeta)}{w_\zeta(\zeta)}. \quad (4)$$

At infinity ($z \to \infty$) or at point $D(\zeta=id)$, the following conditions should be imposed:

- for the given velocity

$$w_{\zeta}(id) = V_e e^{i\alpha} \text{ with } V_e = \frac{1}{\sqrt{1+\sigma}}, \quad (5)$$

- and the single value of the function $z(\zeta)$ at point $D(id)$:

$$\oint_{\zeta=id} z_\zeta(\zeta) d\zeta = 0. \quad (6)$$

The unknown parameters $a$, $b$, $c$, $d$ can be uniquely determined for the given angle of attack ($\alpha$) and cavitation number ($\sigma$).

Some of the calculated results are represented in Figures 4 through 7. Calculations confirm that the partial cavitating flow of a given foil can be achieved for the cavitation number $\sigma \geq \sigma_{\min}$. The minimum possible $\sigma$ should be calculated for each foil. Figure 4 shows the cavity shape on a flat plate for short length of the cavity. Another regime of flows for the same $\alpha$ and $\sigma$ is shown in Figure 5. For the comparison, the cavity shape corresponding to end-flat plate model is depicted. As shown in Figure 5, both models give practically the same results.

![Figure 4. The cavity shape for partial cavitation](image)

![Figure 5. Comparison of cavity shapes corresponding to single spiral model and end-flat models (fill area)](image)
cavitating flow with smooth closure of the cavity end could be realized.

![Figure 6](image)

Figure 6. Cavity shape for the foils with the trailing wedge.

Results represented here are evidence of the complicity and multiversions of partial cavitation. Because of the infinite multisheeted surface, it is quite impossible to apply this model or the model with a re-entrant jet to the investigation of partially cavitating hydrofoils with curvilinear boundaries. Furthermore, it is difficult to apply the cavitation model having a singularity at the cavity end to the calculation of cavitating flow by using finite-difference numerical methods. For those purposes, it is quite convenient to apply the cavitation model having an end plate.

![Figure 7](image)

Figure 7: The plate with inclined upper side and with smooth closure of the cavity end.

3. GRAVITY EFFECT IN CAVITATION

Many authors investigated the gravity effect in the cavitating flow. After the introduction of linear cavity theory by M.Tulin (1953), numerous problems of cavitating flow have been investigated that address the effect of gravity (Acosta 1961, Galanin A.V., Gusev V.A. 1978, Efremov I.I. 1978). The non-linear problem of gravity effects has been considered using different cavity models. These include Lenau 1963 – a double rectilinear model in a longitudinal gravity field; Larock and Street 1967 – Tulin’s model in a longitudinal gravity field; Vishnevsky, Kotlyar and A.G. Terentiev 1974 – a double rectilinear model in an inclined gravity field; Kotlyar and Troepolskaya 1975 – a re-entrant double rectilinear model in a transverse gravity field.

![Figure 8](image)

Figure 8: A cavitating wedge in a longitudinal gravity field: (a) flow sketch in the \( z = x + iy \) plane, (b) parametric \( \zeta = \xi + i\eta \) plane.

In this section, the value problem of cavitating flow around a wedge in a longitudinal gravity field (Figure 7a) is considered.

3.1. Formulation of the value problem

Consider a 2-D steady cavitating flow past a wedge, as shown in Figure 8a. Assuming an inviscid, incompressible fluid and irrotational flow, one can define the complex potential as the analytic function \( w = \phi + i\psi \), so that the complex velocity \( w = \frac{dz}{dw} \). In order to determine \( w(z) \), the following boundary conditions are proposed:

\[ \psi = \text{const} \quad \text{or} \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{(7)} \]

and on the cavity shape -

\[ \left| \frac{dw}{dz} \right|^2 + gy = 1. \quad \text{(8)} \]

Furthermore, the function \( \omega(w) = \ln(\frac{dw}{dz}) \), as a solution of the mixed value problem in general, has at point \( (w_0) \) a singularity of the type \( \omega(w) \approx (w - w_0)^{-1/2} \).

Equation (8) is written in non-dimensional form, so that the value of the speed at detachment point \( A \) or \( B \) is unity.

The complex velocity should satisfy the supplementary condition at infinity

\[ w(z) \big|_{z \to \infty} = -IV_0. \quad \text{(9)} \]
Furthermore, at infinity the stream should be undisturbed, i.e., the wedge and the cavity could be enclosed by a continuous closed curve. This case can be written mathematically as

\[ \oint_C dz = 0. \]  

(10)

In order to uniquely determine of the solution, the following dimensionless parameters are given:

- The cavitation number
  \[ \sigma = 2(p_\infty - p_c)/\rho V_\infty^2 \]  
  (11)
  where \( p_\infty \) is the pressure at infinity along the x-axis \((y = 0)\), \( p_c \) is the cavity pressure, \( V_\infty \) - is the stream velocity at infinity, \( \rho \) - is the density of the fluid.

- The Froude number
  \[ Fr = V_\infty^2/gL \]  
  (12)
  where \( g \) is the acceleration of gravity, \( L \) is the length of a wedge side.

### 3.2. Parametric solution

The value problem formulated above can be solved parametrically by conformal mapping the physical \( z \)-plane onto the half unit circle of the auxiliary \( \zeta \)-plane as shown in Figures 8a and 8b respectively. The transformation from the \( w \)-plane to the \( \zeta \)-plane can be written in differential form as:

\[ \frac{dw}{d\zeta} = w_\zeta(\zeta) \]  

(13)

Instead of the complex velocity, it is preferable to solve its logarithm function

\[ \theta = \ln w_\zeta = \ln v - i\theta \]  

and

\[ \zeta(1 - \zeta^2)/(\zeta^2 + a^2)^2(a^2\zeta^2 + 1)^2 = \tau \]  

(14)

The mapping function \( z(\zeta) \) is determined from the differential equation

\[ \frac{dz}{du} = w_\zeta e^{-i\omega(\zeta)}. \]  

(15)

Equations (14) and (15) are the general solution of above-mentioned problem. They include unknown parameters \( a \) and \( b \) and the undetermined coefficient \( C_n \). Conditions (3) and (4) produce two real equations, which can determine \( a \) and \( b \) for given coefficients \( C_n \). Namely for \( C_n = 0 \), equations (5) to (7), together with conditions (3) and (4), uniquely determine the solution to the problem of flow past a wedge in weightlessness.

### 3.3. Determining the coefficients \( C_n \)

The real and imaginary parts of equation (6) on the arc \( \zeta = e^{it} \) are defined as:

\[ \ln v = \sum_{n=1}^{\infty} C_n(\cos 2n\tau - 1) \]  

(16)

\[ \theta = \pi + \mu - \mu \tan \tau - \sum_{n=1}^{\infty} C_n \sin 2n\tau \]  

(17)

Condition (2) can be written in differential form as

\[ \frac{v^2}{g} \frac{d\ln v}{d\tau} + \frac{dv}{d\tau} = 0. \]  

(18)

By using Equations (12) to (17), one obtains:

\[ \sum_{n=1}^{\infty} C_n 2n \sin 2n\tau = g \sin 2\tau (a^2 + 1)^2 e^{-\sum_{n=1}^{\infty} C_n(\cos 2n\tau - 1)} \sin \theta(r). \]  

(19)

For a given \( g \), the coefficient \( C_n \) can be found by using either the collocation approach or by an iterative process. Multiplying Equation (19) by \( \sin 2n\tau \) and integrating over the interval \([0, \pi]\), one obtains:

\[ C_n = \frac{g}{n^2} \frac{\sin 2\tau (a^2 + 1)^2 e^{-\sum_{n=1}^{\infty} C_n(\cos 2n\tau - 1)} \sin \theta(r) \sin 2n\tau d\tau}{(a^2 + 1)^4}. \]  

(20)

The last equation is used for the iteration.

### 3.4. Numerical examples

Cavity shapes for a wedge with \( 2\pi\mu = 30^\circ \) and for a flat plate are shown in Figure 9. Negative Froude numbers correspond to coincidently directed speed and gravity.
Calculations show that the cavities for certain $\sigma$ and $Fr$ have a closed boundary with a cuspidal point at end of the cavity, which was determined first in (Acosta 1961) and then in (Lenou 1963). The lift for that case coincides with water buoyant force (Archimedes force). For negative Froude number, the lift can be zero for certain $\sigma$ and $Fr$ (point-line in Figure 9).

4. INCLINED PENETRATION [ENTRY] OF A FOIL INTO A LIQUID

The penetration of a body into a liquid represents one of the more complicated problems in hydrodynamics. Only a few cases of this problem have been investigated at present; for the most part investigations have been limited to symmetric entry (see references by Sagomonyan 1974, Korobkin & Pukhnachov 1988, Terentiev 2002).

B. Yim (1971) has investigated the non-symmetric but perpendicular entry of thin foils. Two approaches to solving the problems of inclined penetration of thin foils were considered by A.G. Terentiev (1977, 1979). The first of them is based on complex velocity, the second on the linear value problem presented in Equations (21) to (25).

4.1. Complex velocity approach

Consider the motion of a partially submerged foil (Figure 10a). Let the abscissa axis be oriented towards the surface of the body is imposed as:

$$\text{Figure 10: Inclined entry in a liquid: (a) flow sketch in the } z = x + iy \text{ - plane, (b) parametric } \zeta = \xi + i\eta \text{ - plane.}$$

The linear value problem is formulated on the half plane with an inclined slit (the definitional domain in Figure 10a is designed by a dashed-line).

The linear kinematical condition on the wetted surface of the body is imposed as:

$$\text{Im}(\frac{\partial w}{\partial \zeta}) = g(z,t), x \in (l,0), \text{ and } y = \pm 0 \quad (21)$$

where $g(z,t)$ is described by the given motion of the body.

The pressure is determined by the linear Bernoulli equation as

$$p = -\rho \text{ Re}(\frac{\partial w}{\partial t}) \quad (22)$$

so that the dynamic condition on the inclined dashed-line is described by

$$\text{Re}(\frac{\partial w}{\partial \zeta} e^{-i\eta}) = 0 \text{ on } \text{Im}(z e^{i\eta}) = 0 \quad (23)$$

A solution of the mixed value problem should possess a singularity of the type

$$\frac{\partial w}{\partial \zeta} = (z - l)^{1/2} \quad (24)$$

and a zero of the second kind at infinity, that is,

$$\frac{\partial w}{\partial \zeta} \approx z^2, \quad z \to \infty \quad (25)$$

Solving the above formulated mixed value problem (20) to (24), one conformally maps the definitional domain in the $z$-plane onto the upper half-plane of the auxiliary $\zeta$ - plane (Figure 10b). The transformation is:

$$z = l(1-\zeta)^{1/2}, \quad a = 1 - \gamma \quad (26)$$

Instead of the function $w_c(z,t)$, it is more convenient to solve another value problem of the partial derivative

$$\frac{\partial w}{\partial \zeta} = w_c(\zeta,t),$$

which satisfies the following conditions:

$$\text{Re } w_c = 0 \text{ on } \zeta \in (-\infty,-a) \text{ and } \zeta \in (1, \infty), \quad \eta = 0 \quad (27)$$

$$\text{Im } w_c = g(x(\zeta,t),t) x_\zeta (\zeta,t) \text{ on } \zeta \in [-a,1], \quad \eta = 0 \quad (28)$$

The solution of the value problem is of the form

$$w_c = \frac{a^{\alpha-1}}{\pi \sqrt{(1-\zeta)(a+\zeta)}} \int_1^{\alpha} \frac{g u d u}{\tau - \zeta} \quad (29)$$

The complex velocity $w_c = w_c(\zeta,t)/z_c(\zeta,t)$ is a function of the variable $\zeta$, and of the time $t$ and together with Equation (26), they represent the solution of the value problem presented in Equations (21) to (25).

Since the variable $\zeta$ is the function of $z$ and $t$, then it is quite complicated to obtain the complex potential $w(z,t)$ and its partial time derivative $\frac{\partial w(\zeta,t)}{\partial t}$. On the surface of the body, the potential $\varphi(\zeta,t)$ can be determined by integration of Equation (28) over the interval $(-a, \zeta)$, and then the pressure can be expressed in the form:

$$p = -\rho \left( \frac{\partial \varphi(\zeta,t)}{\partial t} + \frac{\partial \zeta(x,t)}{\partial t} \frac{\partial \varphi(\zeta,t)}{\partial \zeta} \right) \quad (30)$$

where

$$\frac{\partial \zeta(x,t)}{\partial t} = \frac{l(1-\zeta)(a+\zeta)}{l\zeta} \quad (31)$$

All other characteristic curves of the problem can be calculated by using Equations (26) to (30). Indeed, all expressions in general have single and double integrals. Only for a wedge with inclined sides and perpendicular entry can simple expressions be obtained for the cross force, the longitudinal force, the leading edge force and the moment about origin of coordinates ($O$). These are respectively:

$$Y = \rho l(U - 2U^2)/(a_1 + a_2)/\pi \quad (32)$$

$$X = \rho l(U - lU^2)[0.5(a_1 + a_2)^2 + (a_1 - a_2)^2 \ln 2] / \pi \quad (33)$$

$$X_0 = -\rho l U^2(a_1 + a_2)^2/2\pi \quad (34)$$

where $a_1$ and $a_2$ are inclined angles of the wedge sides: the equality $a_1 = -a_2$ corresponds to a wedge being symmetric about the x-axis and $a_2 = a_1$ corresponds to a flat plate.
Note, that the expression in Equation (33) differs from other authors (Wagner 1932, Yim B. 1971, Sagomonyan 1974), whose work has some inaccuracies. Calculated data are depicted in Figures 11 and 12. Figure 19a shows the lift coefficient \( C_L = 2L/\rho V^2 \alpha \), the drag coefficient \( C_D = 2D/\rho V^2 \alpha \), and the center of pressure \( x_p = M/L \) of a flat plate \( \alpha = -\alpha_1 = -\alpha_2 \) for an inclined entry with constant speed \( V = \text{const} \) as functions of \( \gamma \). The time-dependencies of the same dynamic characteristics for \( \gamma = 1/4 \), unit acceleration and unit initial speed are shown in Figure 12. Similar curves for a wedge, that is, \( \alpha \alpha \alpha \alpha = \alpha \alpha \alpha \alpha \), are shown in Figure 20.

![Figure 11](image1.png)

**Figure 11.** Lift \( C_L \) and drag \( C_D \) coefficients and the center of the pressure \( x_p \) of a flat plate and a wedge for inclined entry: Frames a and c show the dependencies on inclined angle, and frames b and d show the dependencies on time.

### 4.2 Partial time-derivative

The approach above is quite difficult to use in the investigation of many other problems connected with unsteady flow with free boundaries because of the integration and partial differentiation of very complex functions. It is preferable to formulate a value problem for the partial derivative with respect to time \( w_l(z,t) = \partial^2 w(z,t)/\partial t \), the real part of which determines the pressure

\[
p = -\rho \varphi_t + \text{const}.
\]

Using partial derivatives \( w_l \) and \( w_r \), one can also analytically investigate the non-linear problems of unsteady moving bodies in a liquid (Terentiev 1981, 1989). Below is an example of using this approach to solve unsteady linear problems.

The boundary condition (20) can be rewritten by integrating along the x-axis

\[
\psi(x,t) = \frac{1}{\alpha(t)} \int g(x,t) dx + T(t)
\]

Hence,

\[
\varphi_x = s g(x,t) - \frac{1}{\alpha} \int g(x,t) dx + \tilde{T}(t).
\]

where \( \tilde{T} \) is to be found.

The dynamic condition on the free boundary is

\[
\varphi_t = 0
\]

Therefore, the function \( w_l(z,t) \) should be a solution of a mixed value problem of conditions (36) and (37). The solution of a mixed value problem has a certain singularity at the leading edge, but the stream function should be uninterrupted at that point. The condition of continuity of a stream function at the leading edge develops an integral equation for unknown function \( T(t) \).

### 4.3 Entry of a flat plate

Consider the inclined entry of a flat plate with a ventilated cavity and constant velocity (Figure 12a).

![Figure 12](image2.png)

**Figure 12.** Entry of a plate with a ventilated cavity: (a) physical z-plane, (b) parameter \( \xi \)-plane.

On the slit bank \( OB \), which has a variable length \( l(t) \), the cinematic condition is \( \psi_x = -\varphi_t = l \alpha \). Due to Equation 36, the stream function on the plane is of the form:

\[
\psi = \tilde{T} - i \alpha(x - l) + T(t).
\]

Hence, the boundary conditions for the partial time-derivative \( w_l = \varphi_t + i \psi_x \) on the \( \xi \)-axis are expressed as follows:

\[
\psi_x = \tilde{T} - l \alpha \quad 0 \leq \xi \leq 1 \quad \text{and} \quad \varphi_t = 0 \quad \text{outside} \quad (0, 1)
\]

At the point \( O \), the function \( w_l \) has a singularity of the form \( \xi^{-1/2} \), is limited at point \( B \), and approaches zero in infinity so that it is determined by:

\[
w_l = i(\tilde{T} - l \alpha) \left( 1 - \frac{\xi - 1}{\sqrt{\xi}} \right).
\]

The same function on the \( \xi \)-axis determines the partial time derivative of the stream function outside \( (0, 1) \), so that at any moment the stream function itself can be obtained by integration with respect to time over the interval \( (0, t) \). Equating with function for \( T(t) \), one obtains an integral equation of the form
\[
\int_{0}^{\tau} \left( \dot{\zeta} + i\alpha \right) \frac{\zeta - 1}{\zeta} \, d\tau = -i t^2 \alpha. \quad (41)
\]

The variable \( \zeta \) is a function of times \( t \) and \( \tau \) as a result of the equality
\[
s(t) = s(0)(1 - \zeta)^{1/2} \left( 1 - \frac{\zeta}{a} \right)^{1/2}. \quad (42)
\]

For constant speed \( V = \dot{l} = \text{const} \), the solution of Equation (41) and all the hydrodynamic characteristics can be expressed in analytic form (Terentiev 1979).

Figure 13 shows the dependencies of the lift coefficient \( C_L \) and the center of the pressure \( x_0 \) of a plate on the angle of inclination. For \( \gamma = 1 \), the lift coefficient \( C_L = \pi / 2 \) and the center of the pressure \( x_0 = 3 / 4 \). This result coincides with those in the previous section for a wedge when the angle of inclination approaches zero.

\[\begin{align*}
\text{Figure 13. Dependencies of the lift coefficient } (C_L) \text{ and the center of the pressure } (x_0) \text{ of a plate on the angle of inclination.}
\end{align*}\]

### 4.4. Entry of a finite wedge

Consider the vertical entry of a finite wedge into a liquid as in (Figure 14).

\[\begin{align*}
\text{Figure 14. Vertical entry of a wedge.}
\end{align*}\]

The sides of the wedge are equal and form the angles \( \alpha_1 \) and \( \alpha_2 \) with the x-axis. The wedge moves with constant velocity, \( V \).

The boundary condition are expressed as:
\[
\psi = \begin{cases} 
-V^2 \alpha_1 + \bar{T}, & \xi \in (-b, 0) \\
-V^2 \alpha_2 + \bar{T}, & \xi \in (0, b) 
\end{cases}
\]

\[\begin{align*}
\varphi = 0 \text{ outer interval } (-b, b). \quad (43)
\end{align*}\]

The value problem can be solved analytically in the form
\[
w = -\frac{V^2 \beta}{\pi} \ln \left( \frac{\sqrt{\zeta^2 - b^2} + ib}{\zeta} \right) + i V^2 \alpha \left( 1 - \frac{\sqrt{\zeta^2 - b^2}}{\zeta} \right) \quad (44)
\]

where \( \alpha = (\alpha_1 + \alpha_2) / 2 \), \( \beta = \alpha_2 - \alpha_1 \), \( \mu = \bar{T} / V^2 \alpha - 1 \). \quad (45)

Two stages of the entry of the wedge should be considered: one without and one with the cavity ventilated. Wolterra’s types of integral equation for unknown function \( \mu(t) \) can be solved analytically for the first stage (\( \mu = -\pi / 2 \)). The results coincide with those obtained in section 4.1. For the second stage, the integral equation is expressed of the form
\[
\int_{t_0}^{t} \mu(t) \left( \frac{\tau - t_0^2}{\tau^2 - t_0^2} \right) dt = -t \left( 1 - \frac{2}{\pi} \arcsin \frac{t_0}{l} \right) \quad (46)
\]

where \( t_0 \) is the time of the entry of the whole wedge.

Equation (46) can be solved only numerically (Terentiev 1979). When the distance \( s \) between the wedge and the free surface of liquid approaches infinity as \( s \to \infty \), all hydrodynamic parameters approach the limits corresponding to Kirchhoff's Model: i.e., the Lift \( L = \pi \sqrt{V} \rho \), the Drag \( D = -\rho V^2 \beta^2 l / \pi \), and the center of the pressure \( \lambda = 1 / 4 \).

Figure 15 shows dependencies of ratios \( L / L_\infty \), \( D / D_\infty \) and \( \lambda / \lambda_\infty \) on an inverse value of the distance \( (l/s) \).

\[\begin{align*}
\text{Figure 15. Dependencies of the hydrodynamic parameters on the depth of submergence.}
\end{align*}\]

### 5. UNSTEADY MOTION OF FOILS WITH KIRCHHOFF’S CAVITY

It is well known that the study of unsteady cavitating flow in a plane has difficulty concerned with a condition at infinity. It is impossible to satisfy both a finite pressure at infinity and at the same time have closure of the cavity. But some facts of unsteady cavitating flows can be established by examination Kirchhoff’s model with a
semi-infinite cavity. Below impact move of the flat plate with such cavity is studied.

Consider the motion of a flat plate with Kirchhoff’s cavity attaching at the trailing edge and at the same point on the suction surface (Figure 16).

![Figure 16. Sketch of flat plate motion (a), and a parametric plane (b).](image)

The plate is of unit length, the distance of the leading edge from origin of non-moving coordinates is designated as \( s(t) \), the length \( AB \) is equal to \( a \), the angle of attack \( \alpha \) is small. The speed of the plate is given as:

\[
\begin{align*}
V_1 &= \begin{cases} 
= 0, & t < t_{\text{const}} \\
= V_0, & t \geq t_{\text{const}}
\end{cases} \\
V_2 &= \begin{cases} 
= 0, & t > t_{\text{const}}
\end{cases}
\end{align*}
\]

The potential and its partial time-derivative satisfy the following boundary conditions:

On the wetted plate surface:

\[
\psi = T(t) + \alpha \psi (s - a), \quad \psi_0 = \bar{T} - \alpha \psi^2
\]

On the cavity surface:

\[
\phi = 0, \quad \phi_0 = 0
\]

To solve the problem, it is divided into three value problems: a steady flow with constant speed \( V_0 \), an impact problem for the increment of the speed \( V_0 - V_1 = \Delta V \) and finally non-stationary problem of moving with constant speed but taking into account the history of changes in the stream function. To solve all these value problems, the flow domain in the physical plane must be mapped onto the upper half-plane of the auxiliary plane (Figure 16b) by the transformation

\[
\zeta = i \sqrt{z - a}
\]

Leaving out the intermediate transforms (see Terentiev and Mihailov V.M. 1979), we give the final integral equation of the Volterra type for the function \( \mu(s) \)

\[
\int_{0}^{s} \frac{\mu(s')}{\sqrt{s - s'}} K(s, s') ds' = F(s),
\]

where

\[
K(s, s') = \sqrt{(1 + s - s')(a + s - s') + \sqrt{a + s - s'}}
\]

and

\[
F(s) = \frac{2 \sqrt{(1 + s + a) (a + s)}}{2 \sqrt{(1 + (s + a) (a + s)) + \sqrt{a + s}}}
\]

The function \( \bar{T}(s) \) is related to \( \mu(s) \) by the equality

\[
\bar{T} = V_2^2 \alpha \left( 1 - \frac{V_1}{V_2} - \frac{1 - V_1}{V_2} \mu \right)
\]

The lift and drag coefficients are calculated as

\[
C_L = 2 \pi \alpha c \mu^*, \quad C_D = 2 \pi \alpha c \mu^* \left( \mu^* \sqrt{a - c^2} \right)
\]

where \( \mu^* = V_1/V_2 + (1 - V_1/V_2) \mu, \quad c = (1 + \sqrt{a})/2 \).

For \( a = 1 \), Equation (51) converts to the Wagner equation problem of impact motion of a thin foil without any cavities (Wagner 1925):

\[
\int_{0}^{s} \frac{1 + s - s'}{\sqrt{s - s'}} \mu(s') ds' = \sqrt{s(s + 1)}
\]

which has an approximate solution as obtained by V.V. Mihailov:

\[
\mu(s) = \frac{1 + s}{2 + s}
\]

Note that Wagner and subsequent authors considered the problem by using a complex velocity and vertical layer past the foil. The integral equation differs from (54); its right-hand member holds an integral. Moreover, hydrodynamic expressions are rather more complicated than equation (53).

On substituting \( a = 0 \), equation (51) yields the integral equation for the Kirchhoff model with attachment point at leader edge (B)

\[
\int_{0}^{s} \frac{1 + s - s'}{\sqrt{s - s'}} \mu(s') ds' = \frac{s^{1/2}}{2 \sqrt{s + 1}}
\]

The solution of the last equation permits an asymptotic expansion as:

\[
\mu(s) = \frac{\gamma}{2 \sqrt{s}} - \frac{9 - 4 \pi^2}{16} + \frac{\gamma (33 + 4 \pi^2)}{16} \sqrt{s} + ...
\]

where \( \gamma = \Gamma(5/4) \Gamma(3/4) \sqrt{\pi} \approx 0.4173 \), \( \Gamma(z) \) is Gamma-function.

The function (57), as well as the lift, approach infinity as the distance approaches zero. This implies that the attachment point cannot coincide with the leading edge but rather at a certain point on the upper surface of the plate at impact moment. But this needs an experimental confirmation.

![Figure 17. Dependence of the solution \( \mu \) on the distance and the ratio \( 1/s \).](image)

The dependence of \( \mu(s) \) is shown in Figure 17. The lift depends on the distance \( s \) and similarly on length \( a \).
6. CROSSING THE FREE SURFACE OF TWO FLUIDS BY A WEDGE WITH AN ATTACHED CAVITY

Consider the perpendicular crossing of the free boundary between two liquids of densities, \( \rho_1 \) and \( \rho_2 \), by a wedge with an attached Kirchhoff cavity (Figure 18). The distance of the wedge from free boundary is \( s(t) \). The half angle opening is such that \( \alpha (\alpha \ll \alpha) \). The length of the wedge is assumed to be unity.

![Figure 18: Crossing a free boundary by a wedge with a cavity.](image)

Because of the symmetry about x-axis, the value problem can be formulated for upper half-plane. The flows on each side of the free boundary are determined by two different potentials: \( w_1(z,t) \) and \( w_2(z,t) \). Three positions of the wedge should be considered: (a) the wedge is located only on the right side of the free boundary, (b) it intersects the free boundary, and (c) it moves beyond the intersection to the left side of the free boundary (Figures 18a, b, and c). The partial time-derivatives for case-a satisfy the following conditions:

\[
\text{Im}(\partial w_1/\partial t) = \begin{cases} 
  \int (f(x-s) - s^2 f'(x-s), s - 1 \leq s, y = 0 \\
  0, \quad 0 \leq x \leq s - 1, y = 0 
\end{cases} \\
\text{Re}(\partial w_1/\partial t) = 0, s \leq x < \infty, y = 0
\]

(58)

where the function \( y = f(x-s) \) determines the wedge size, and \( f'(x-s) \) is the derivative with respect to its argument.

Further, the pressure on both sides of free boundary is the same, i.e., the following equalities on the vertical or y-axis should be satisfied:

\[
\rho_1 \text{Re}(\partial w_1/\partial t) - \rho_2 \text{Re}(\partial w_2/\partial t) = 0 \\
\text{Im}(\partial w_1/\partial t) - \text{Im}(\partial w_2/\partial t) = 0
\]

(60)

Boundary conditions for the other cases, (b) and (c), are imposed likewise.

The first equality, Equation (60), can be fulfilled by an additional function introduced as:

\[
W_i = \begin{cases} 
  w_{1i}(z), \text{Re} z \geq 0 \\
  Aw_{2i}(z) + Bw_{2i}(-\bar{z}), \text{Re} z \leq 0
\end{cases}
\]

(61)

where

\[
A = \frac{2\rho_2}{\rho_1 + \rho_2}, \quad B = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}
\]

If \( \beta(x) = \psi_{\beta} \) is known such that \( s \leq x < \infty \), then the function \( W_i \) can be obtained from Cauchy value problem and functions \( w_{1i} \) and \( w_{2i} \) can be determined from Equation (60). Using the second equality in (57), the unknown \( \beta(x) \) can be obtained from integral equation of the form:

\[
\beta(x) + \frac{B}{\pi} \int_{s}^{\infty} \frac{\beta(x')dx'}{(x' + x)(x' + s(x' + s))} = F(x,s,B)
\]

(62)

where \( F(x,s,B) \) is a calculated function.

The integral in Equation (62) has been calculated numerically by transformation to the integral over the interval \((0,\pi)\) and by expansion of the function \( \beta(x) \) using the trigonometric series.

The calculated data for a wedge with straight sides crossing the surface between two fluids are presented in Figures 19 and 20. Figure 19 shows the dependence of the drag coefficient \( C_D = 2D / \rho V_0^2 \alpha^2 \) on the distance \( s/l \) for \( B = 1, 0.5, -0.5, -1 \).

![Figure 19. Drag coefficient dependence on s/l](image)

The cavity shapes produced when crossing the free boundary of two liquids are depicted in Figure 20. It is shown that the cavity boundary breaks up at the free surface.

![Figure 20. Cavity shapes when crossing a free surface.](image)
boundary. Apparently, in a real process, vortices or some other phenomena such as jets and bubbles are produced.

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