SOLUTION OF AXISYMMETRIC PROBLEM OF CAVITY FLOW PAST SPHERE BY THE RYABUSHINSKII SCHEME WITH HELP OF INTEGRAL TRANSFORMATION

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ABSTRACT

The problem of cavity flow past a sphere by the symmetric Ryabushinskii scheme is under consideration. The fluid velocity is constant on the free surface.

The solution of the axisymmetric problem is reduced to solution of some plane one to determine the function of a complex variable \( f(z) \). The function is analytical in a domain with boundaries shape coinciding with shape of the boundaries of flow meridian cut.

The potential and the stream function of the axisymmetric flow are sought by the Pologii integral transformation. The auxiliary plane problem are written in the form of integral equations. The auxiliary plane flow determining by the analytical function \( f(z) \) must have penetrable boundaries. The law of penetrability is given as the equations of impenetrability of boundaries of the axisymmetric problem.

The function \( f(z) \) is constructed as a cubic spline. The spline interpolates the dependence of the real part \( f(z) \). The dependence of the imaginary part \( f(z) \) is found by the Schwartz's formula.

Due to analyticity of integrands we may integrate along any curve interior to the flow or along the boundary. And also we may map conformally flow region of \( z=\infty \) into a unit disc, for example, so that the circle corresponds to the body bound. Then taking integral along the circle we have

\[
\Phi(x_0, r) = -\frac{2}{\pi} \mathrm{Im} \int_{z_0}^{x_0 + ir} dz \frac{dW}{dz} g(z) + V_\infty x_0, \tag{1}
\]

\[
\Psi(x_0, r) = \frac{2}{\pi} \mathrm{Im} \int_{z_0}^{x_0 + ir} dz \frac{dW}{dz} g(z) + V_\infty, \tag{2}
\]

where \( z_0 = x_0 + ir \) is interior point of the flow, \( x_1 + i0 \) is some point on the symmetry axis \( x \); \( \overline{z}_0 = x_0 - ir \).

\[
\Psi(x_0, r) = -\frac{2}{\pi} \mathrm{Im} \int_{z_0}^{x_0 + ir} dz \frac{dW}{dz} \frac{d\sigma}{g(z)} + V_\infty x_0, \tag{3}
\]

\[
\Psi(x_0, r) = \frac{2}{\pi} \mathrm{Im} \int_{z_0}^{x_0 + ir} dz \frac{dW}{dz} \frac{d\sigma}{g(z)} + V_\infty, \tag{4}
\]

1. INTRODUCTION

Let’s consider the plane problem of flow past some profile symmetric about the \( x \)-axis on the plane \( z=x+iy \). The complex potential is \( W(z)+V_\infty z \), where \( V_\infty \) is the fluid velocity at the infinity directed along the \( x \)-axis. Simultaneously let’s consider the axisymmetric problem of flow past a body with the shape of meridian cut similar to the profile. The coordinate systems of the plane and axisymmetric flows are combined so that the \( x \)-axis is common and the axis \( y \) and \( r \) coincide (fig. 1).

The potential \( \Phi \) and the stream function \( \Psi \) of the axisymmetric field are expressed in terms of the analytical function of a complex variable \( W(z) \) by the formulas (the modified Pologii transformations [1], [2]):

\[
\Phi(x_0, r) = -\frac{2}{\pi} \mathrm{Im} \int_{z_0}^{x_0 + ir} dz \frac{dW}{dz} g(z) + V_\infty x_0, \tag{1}
\]

\[
\Psi(x_0, r) = \frac{2}{\pi} \mathrm{Im} \int_{z_0}^{x_0 + ir} dz \frac{dW}{dz} g(z) + V_\infty, \tag{2}
\]

The values of the multifunction \( g(z) \) are defined as

\[
g(z) = \left| z - z_0 \right|^{3\pi \over 2} e^{i(\arg(z-z_0)+\arg(z-z_0))\over 2}. \tag{3}
\]

Assume the argument of the differences \( z-z_0 \) and \( z-\overline{z}_0 \) varied in \( \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right] \). It is similar to drawing cuts from the points \( z_0 \) and \( \overline{z}_0 \) vertically downwards as in fig.1. In this case the function \( g(z) \) is odd for the parameter \( z-x_0 \) and analytical in the whole plane \( z \) with exception of the point \( z=\infty \) and intercept of straight line between the points \( z_0 \) and \( \overline{z}_0 \).

Due to analyticity of integrands we may integrate along any curve interior to the flow or along the bound. And also we may map conformally flow region of \( z\)-plane onto a unit disc, for example, so that the circle corresponds to the body bound. Then taking integral along the circle we have

\[
\Phi(x_0, r) = -\frac{2}{\pi} \mathrm{Im} \int_{\sigma_0}^{\sigma_1} d\sigma (x_0) \frac{d\sigma}{g(z)} + V_\infty x_0, \tag{3}
\]

\[
\Psi(x_0, r) = \frac{2}{\pi} \mathrm{Im} \int_{\sigma_0}^{\sigma_1} d\sigma (x_0) \frac{d\sigma}{g(z)} + V_\infty, \tag{4}
\]
The values of velocity components are found from (3), (4).
\[ V_x = \frac{\partial \Phi}{\partial x} = -2 \text{Re} \int_0^\infty \frac{d^2W}{dz}\sigma \left( z - z_0 \right) d\sigma \left( z - z_0 \right) + V_\infty, \]
(5)
\[ V_r = \frac{\partial \Phi}{\partial r} = -2 \text{Re} \int_0^\infty \frac{d^2W}{dz}\sigma \left( z - x_1 \right) d\sigma \left( z - x_1 \right). \]
(6)

Let us consider the limiting value of the integral (1) as \( z_0 \to x_1 + i0 \). In this case the function \( g(z) \to z - x_1 \) at the point \( z = x_1 \) the integrand will have a simple pole with the principal part \( \frac{dW}{dz}(x_1 + i0)/(z - x_1) \). Let the contour of the axisymmetric body be located to the right of the point \( z = x_1 \), (Figure 1). If in the upper half-plane of \( z \) (Im \( z > 0 \) ) outside the body boundary we consider closed contour to the left of the straight line \( (x_1 + i0, x_1 + i\infty) \), the integral along which is equal to zero. Then by deforming this contour we can obtain form for the same value of the velocity potential
\[ \Phi(x_1,0) = \frac{dW}{dz}(x_1,0) + V_\infty x_1. \]
(7)

Analogous formula can be found for a point \( x_2 \) situated on the right of the axisymmetric body
\[ \Phi(x_2,0) = -\frac{dW}{dz}(x_2,0) + V_\infty x_1. \]
(8)

Near the critical point \( x_0 \) the cone-shaped vertex of the body the potential is equal to \( \Phi(x,0) - \Phi(x_0,0) = O(x - x_0)^\gamma \), where the exponent \( \gamma \) depends only on the angle of taper. Consequently, we get
\[ \frac{dW}{dz}(z) - \frac{dW}{dz}(x_0) = V_\infty (z - x_0) + O[\varepsilon^\gamma], \]
where the upper sign corresponds to the left and the lower sign to the right critical points. For bodies with a smooth surface the exponent \( \gamma = 2 \), i. e., at these points the analyticity of the function \( \frac{dW}{dz}(z) \) does not break down.

The expression (9) for second derivatives takes the form
\[ \frac{d^2W}{dz^2}(z) = V_\infty + O[\varepsilon^\gamma]. \]
(10)

2. STATEMENT OF THE AUXILIARY PLANE PROBLEM

The solution of the auxiliary plane problem is constructed so that the stream function value equal to zero remains on the whole axis with the exception of parts intercepted by the body and the stream function of the plane problem has varied values on the body surface (as for the plane flow past a body with penetrable bounds).

The stream function of the axisymmetric flow and the complex conjugate velocity of the plane flow \( dW/dz \) are related by (4). By virtue of impenetrability of the bound \( AB \) the stream function \( \Psi \) of the axisymmetric problem to have to equal zero. This equation leads to an integral equation, which is the boundary condition for determination of the complex potential of the auxiliary plane problem.

Domains corresponding to the flow in the physical plane \( z \) and in the plane of complex potential \( W \) are conformally mapped onto some canonical domain in the plane \( \zeta \), for example onto a unit disc. The real diameter in the \( \zeta \) plane corresponds to the \( x \)-axis and the circle corresponds to the body surface.

Let’s consider a body symmetric about the straight line (plane) \( x = 0 \). Then the transformation formulas are written in the form
\[ z = L \left[ -\frac{1}{\zeta} + \sum_{m=0}^\infty a_{2m+1} \zeta^{2m+1} \right], \]
(11)
\[ W(\zeta) = L^2 V_\infty \left[ -\frac{1}{\zeta} + \sum_{m=0}^\infty c_{2m+1} \zeta^{2m+1} \right], \]
(12)
where \( L, a_{2m+1}, c_{2m+1} \) are real coefficients. Existence of the factor \( L^2 \) in (12) is necessary to remain of dimension of potential and stream function of the axisymmetric problem.

On the circle \( \zeta = e^{i\theta} \)
\[ \frac{dz}{d\sigma} = i \frac{d\zeta}{d\sigma} = iL \left[ 1 + \sum_{m=0}^\infty (2m + 1)a_{2m+1} \zeta^{2m+1} \right] \]
\[ \frac{dW}{dz} = V_\infty L + V_\infty L \sum_{m=0}^\infty (2m + 1)c_{2m+1} \zeta^{2m+1} \]
\[ + \sum_{m=0}^\infty (2m + 1)a_{2m+1} \zeta^{2m+1} \]
(13)

Then taking into account that the integral (2) of the function \( dW/\zeta \) is identically equal to zero and that the values of the derivative \( dW/\zeta \) need for problem solving, it is convenient instead of (12) to use the expression
\[ \frac{dW}{dz}(\zeta) = V_\infty L \sum_{m=1}^\infty b_{2m} \zeta^{2m}, \]
(14)
\[ \frac{dW}{d\sigma} = iL^2 V_\infty \left[ 1 + \sum_{m=0}^\infty (2m + 1)a_{2m+1} \zeta^{2m+1} \right] \sum_{m=1}^\infty b_{2m} \zeta^{2m}, \]
(15)
\[ \frac{d^2W}{dzd\sigma} = iL^2 \left( \frac{dW}{d\sigma} \right)^{-1} = iL^2 V_\infty \sum_{m=1}^\infty \frac{b_{2m}}{c_{2m+1}} \zeta^{2m}, \]
(16)
\[ \frac{d^2W}{dzd\sigma} = \frac{d^2W}{d\zeta d\sigma} \left( \frac{dz}{d\sigma} \right)^{-1} = V_\infty \left[ \frac{1}{\zeta} + \sum_{m=0}^\infty (2m + 1)a_{2m+1} \zeta^{2m+1} \right]. \]
(17)

Then using the equalities (10), (17) at the critical points \( A \) and \( B \) (\( \zeta = \pm 1 \)) gives
\[ \sum_{m=1}^\infty 2mb_{2m} + \sum_{m=0}^\infty (2m + 1)a_{2m+1} + 1 = 0. \]
(18)

If there are singular points on a plane (angular crest points or points of jet separation from a body surface) presentation of functions in the form of power series leads to low convergence of these series. Power splines are offered to use for such problems solution.
3. SOLUTION BY SPLINE

The flow domain is conformally mapped onto the upper half-plane \( t = \xi + i\eta \) so that the bound of the body corresponds to \([-1,1]\). The solution is found in the form
\[
z(t) = \tilde{L}[f + \tilde{z}(t)],
\]
\[
\frac{dW}{dz}(t) = V_\nu Lf(t).
\]

Note, as \( \zeta = t - \sqrt{t^2 - 1} \) then according to (11) for the body with smooth bound at the point \( A \) near this point
\[
z = L \left[ \zeta + \sum_{m=0}^\infty a_{2m+1} \left( \zeta - i \sqrt{1 - \zeta^2} \right)^{2m+1} \right] = \]
\[
= L \left[ \zeta + \sum_{m=0}^\infty a_{2m+1} \left( \sum_{j=0}^m (-1)^j C_{2m+1}^j (1 - \zeta^2)^j \zeta^{2m+1-2j} \right) - i \sqrt{1 - \zeta^2} \sum_{m=0}^\infty a_{2m+1} \left( \sum_{j=0}^m (-1)^j C_{2m+1}^j (\zeta^2 - 1)^j \zeta^{2m-2j} \right) \right].
\]

For \( \xi \to \pm(1-0) \)
\[
x(\xi) = O(\xi^2 - 1), \quad y(\xi) = O(\sqrt{1 - \xi^2}).
\]

Thus, it is better to approximate by spline the real part of the function \( x(\xi) \).

Let us seek the solution in the nodes \( \xi_m, m=0,\ldots,M, (\xi_0)=1, \xi_M=0 \). Desired parameters will be values \( \text{Re} \tilde{z}(\xi_m) = x_m \) and \( \frac{1}{V_\nu L} \text{Re} \frac{dW}{dz}(\xi_m) = f_m, m=0,\ldots,M \). We find values \( \text{Re} \tilde{z}(\xi) \) and \( \text{Re} \frac{dW}{dz}(\xi) \) in points between nodes with the help of cube spline, which has two continuous derivatives
\[
x(\xi) = \frac{(\xi_m - \xi)^3 - h_m^2 (\xi_m - \xi)^2}{6h_m^2} \alpha_m - \frac{1}{6h_m} + \frac{\xi_m - \xi}{h_m} x_m + \frac{\xi_m - \xi}{h_m} x_m,
\]
\[
where h_m = \sigma_m - 1 - \sigma_m.
\]

Parameters \( d_m \), which are equal to \( S(\sigma) \) first derivative values in nodes, are defined by the solution of linear algebraic system
\[
\alpha_{m-1} + \frac{1}{h_m} + \frac{h_{m+1}}{h_m} \alpha_m + \frac{h_{m+1}}{h_m} \alpha_{m+1} = 6 \frac{X_{m+1} - x_m}{h_m} h_m h_{m+1} - 6 \frac{X_m - x_{m-1}}{h_m} h_m h_{m+1} h_m
\]
for \( m=1,\ldots,M-1 \).

The value \( d_M \) is equal to zero because it is a value of the second derivative of the odd function \( \text{Re} \tilde{z}(\xi) \) for \( \xi=0 \).

The real part of \( -\zeta(t) \sqrt{t^2 - 1} \) is equal to \( x(\xi) \sqrt{1 - \xi^2} \) for \(-1 \leq \xi \leq 1\) and to zero for \( |\xi| \geq 1\) (because \( \text{Im} \tilde{z}(\xi+i0) = 0 \) for \( |\xi| \geq 1 \)). Then \( \tilde{z}(t) \) is sought by the Schwarz formula
\[
\tilde{z}(t) = \frac{1}{\pi \sqrt{t^2 - 1}} \int \, x(\xi) \sqrt{1 - \xi^2} \frac{d\xi}{\sqrt{t^2 - 1}} + \frac{a}{\sqrt{t^2 - 1}}.
\]

For the odd function \( x(\xi) \)
\[
\tilde{z}(t) = \frac{1}{\pi \sqrt{t^2 - 1}} \int \frac{x(\xi) \sqrt{1 - \xi^2}}{\sqrt{t^2 - 1}} \frac{d\xi}{t} + \frac{a}{\sqrt{t^2 - 1}}.
\]

From the condition of boundedness of \( \tilde{z}(t) \) near \( t= \pm 1 \)
\[
a = \frac{2}{\pi} \int \frac{x(\xi) \sqrt{1 - \xi^2}}{\sqrt{t^2 - 1}} d\xi.
\]

The function \( f(t) \) is determined analogously.

4. CAVITY FLOW PAST SPHERE

Boundless stream of ideal incompressible fluid with jet separation and cavity appearance flows past a sphere. The flow is to be potential and symmetric about the plane \( x=0 \) (fig.2).

![Image](image)

Figure 2. Cavity flow past sphere

The velocity at the infinity is equal to \( V_\nu \) and directed along symmetry axis. On the sphere surface
\[
|x-x_c| = R^2,
\]
where \( R \) is the sphere radius, \( x_c \) is abscissa of the sphere center.

On the free surface the following equation fulfills
\[
V_x^2 + V_r^2 = V_\nu^2.
\]

Here \( V_r \) and \( V_r \) are sought by the formulas (5), (6).

Numerically the problem is solved by the collocation method. Here methods described in [2, 3] are used. Impenetrability conditions and the equations (26), (27) are fulfilled at the discrete points \( \xi_m, m=0,\ldots,M \). The nonlinear equation system obtained by this way is calculated over the parameters \( x_m, y_m \) by the Newton’s method with step variation.

The results of problem solution are shown in fig. 3-5. The curves corresponding to the axisymmetric and plane flows are drawn with continuous line and dotted line respectively. We denote
\[
c_x = \frac{2}{\pi \rho V_\nu^2} \int_0^y (P-P_0) dy, \quad \overline{L} = \frac{L}{2R \frac{1+3/Q^2}{1+Q^2}}, \quad \overline{H} = \frac{H}{R \frac{1+2/Q}{1+Q^2}}
\]
for the plane flow;
\[
c_s = \frac{2}{\pi \rho V_\nu^2} \int_0^{2\pi} (P-P_0) 2\pi r dr, \quad \overline{L} = \frac{L}{2R \frac{1+3/Q^2}{1+Q^2}}, \quad \overline{H} = \frac{H}{R \frac{1+2/Q}{1+Q^2}}
\]
for the axisymmetric flow. Here \( R \) is the radius of the cylinder or sphere, \( P \) is the pressure of fluid on the cavitator, \( P_0 \) is the pressure in the cavity, \( L, H \) are the length and width of the cavity (the length of the cavity is the distance \( OO' \), the width corresponds to an ordinate of the point \( F \) ), \( Q = \frac{P_0 - P_0}{\rho V_\nu^2} \) is the
cavitations number, \( \tilde{I}_B = I_B / R \) is relative length of moistened surface of the sphere. The different form of the denominators for \( \tilde{L} \) and \( \tilde{H} \) in (29) and (30) are accepted for convenience of curves disposition in the diagrams. The curves 1-7 in fig.3-5 corresponds to the different cavitations number: 1 – \( Q = 0.01 \), 2 – \( Q = 0.5 \), 3 – \( Q = 1 \), 4 – \( Q = 1.25 \), 5 – \( Q = 2 \), 6 – \( Q = 3 \), 7 – \( Q = 4 \).

The values, which take place under Brilluen condition, are plotted as dash line. These values are extreme on the curves.

![Figure 3](image3.png)  
**Figure 3.** Resistance coefficient dependence on relative length of moistened surface

![Figure 4](image4.png)  
**Figure 4.** Relative cavity length dependence on relative length of moistened surface

Comparison of the curves shows that the cavity length and width for the axisymmetric flow are essentially smaller than for the plane one. Asymptotic behavior of these curves is different for \( Q \to 0 \). This can be explained by the following. The fluid mass deviating by the cavitator in the axisymmetric flow is smaller because of a smaller amount of fluid particles moving near the x-axis.

**REFERENCES**